# Path Integral Formulation for Wave Effect in Multi-lens System

Kazuhiro Yamamoto
*Department of Physical Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*

A formula to investigate wave effect in multi-lens system is presented on the basis of path integral formalism by generalizing the work by Nakamura and Deguchi (1999). Wave effect of a system with two lenses is investigated in an analytic way as a simple application to demonstrate usefulness of the formula and variety of wave effect in multi-lens system.

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## I. INTRODUCTION

Recently several authors have investigated wave effect in gravitational lensing phenomenon, motivated by the possibility that the wave effect might be detected in future gravitational wave observatories [1, 2, 3, 4, 5, 6, 7]. The wave effect can be investigated by solving wave equation, in principle. However, it is difficult to solve the wave equation analytically for general configuration of lenses, excepting the simple case of a single Schwarzschild lens (e.g., [8, 9], see also [10] and references therein, cf. [3]). Indeed, investigation of the wave effect so far is restricted to special case of single lens model with the spherically symmetry, i.e., the Schwarzschild lens model and the singular isothermal sphere lens model (see [1]).

Nakamura and Deguchi developed an elegant formalism for gravitational lens using the path integral approach [6]. The primary purpose of the present paper is to derive a useful formula to investigate the wave effect in general multi-lens system, by generalizing the formalism by Nakamura and Deguchi. We also apply it to a system with two Schwarzschild lenses to demonstrate usefulness of the formula. This paper is organized as follows: In §. 2, we present a generalized formula for multi-lens system. In §. 3, an application of the formula to a simple configuration with two lenses is considered. §. 4 is devoted to summary and conclusion. We use the convention $c = 1$.

## II. GENERALIZED FORMULATION

We start by reviewing the path integral formalism for gravitational lens phenomenon [6]. We consider the Newtonian spacetime with the metric

$$ds^2 = -(1 + 2U(r, \theta, \varphi))dt^2 + (1 - 2U(r, \theta, \varphi))(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{1}$$

where $U(r, \theta, \varphi)$ is the Newtonian potential. Propagation of massless field $\phi$ is described by the wave equation

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi(\mathbf{r}, t) = 0, \tag{2}$$

where $g$ is the determinant of the metric and $\mathbf{r}$ represents the spatial coordinates $(r, \theta, \varphi)$. We consider a monochromatic wave from a point source with the wave number $k$. We set

$$\phi(\mathbf{r}, t) = \frac{A}{r}F(\mathbf{r})e^{-ik(t-r)}, \tag{3}$$

where $A$ is a constant, then Eq. (2) yields

$$\frac{\partial^2 F}{\partial r^2} + 2ik\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2} - 4k^2 U(r, \theta, \varphi)F = 0. \tag{4}$$

Assuming that the first term is negligible compared with the second term and $\theta \ll 1$, Eq. (4) reduces to

$$i\frac{\partial F}{\partial r} = -\frac{1}{2kr^2}\left[\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2 F}{\partial\varphi^2}\right] + 2kU(r, \theta, \varphi)F. \tag{5}$$

Due to analogy of Eq. (5) with the Schrödinger equation, using the path integral formulation of quantum mechanics, the solution can be written as follows,

$$F(r, \theta, \varphi) = \int \mathcal{D}\Theta \exp\left[ik\int_0^r dr\left(\frac{1}{2}r^2\dot{\Theta}^2(r) - 2U(r, \Theta(r))\right)\right], \tag{6}$$

where the dot denotes the differentiation with respect to $r$ and $\Theta$ is used to represent the variables $(\theta, \varphi)$, which are related by $\Theta = (\theta\cos\varphi, \theta\sin\varphi)$. The expression (6) means the sum of all possible path $\Theta(r)$, fixing the initial point (source's position) and the final point (observer's position).

Let us consider multi-lens system, including $n$ lenses as shown in Fig. 1, in which the source is located at the origin of coordinate and an observer is located at $(r, \Theta) = (r_N, \Theta_N)$. The radius between the source and the observer is discretized by $N$ segments, and each discrete radius from the source is labeled by $r_j$ with $j$ from 1 to $N$. Separation neighboring two discrete radii is $\epsilon$. We assume that the $n$ lenses are located at the radius $r = r_{l_m}$ with $m$ from 1 to $n$, and that the thin lens approximation is valid for the lenses. In this case the explicit expression of Eq. (6) is written as Eq. (A2) in Appendix (see also [6]). After some computation, the path integral expression reduces to (see Appendix for details)

$$
\begin{aligned}
F(r_N, \Theta_N) &= \frac{k}{2\pi i}\frac{r_{l_1}r_{l_2}}{r_{l_1,l_2}}\int d^2\Theta_{l_1}\exp\left[ik\left(\frac{r_{l_1}r_{l_2}}{2r_{l_1,l_2}}|\Theta_{l_1}-\Theta_{l_2}|^2-\psi(\Theta_{l_1})\right)\right] \\
&\times \frac{k}{2\pi i}\frac{r_{l_2}r_{l_3}}{r_{l_2,l_3}}\int d^2\Theta_{l_2}\exp\left[ik\left(\frac{r_{l_2}r_{l_3}}{2r_{l_2,l_3}}|\Theta_{l_2}-\Theta_{l_3}|^2-\psi(\Theta_{l_2})\right)\right] \\
&\qquad\qquad\vdots \\
&\times \frac{k}{2\pi i}\frac{r_{l_n}r_N}{r_{l_n,N}}\int d^2\Theta_{l_n}\exp\left[ik\left(\frac{r_{l_n}r_N}{2r_{l_n,N}}|\Theta_{l_n}-\Theta_N|^2-\psi(\Theta_{l_n})\right)\right]
\end{aligned}
\tag{7}
$$

with

$$
\psi(\Theta_{l_m}) = 2\int_{r_{l_m}-\delta r}^{r_{l_m}+\delta r} dr U(r, \Theta), \tag{8}
$$

where $r_{l_{m-1},l_m} = r_{l_m} - r_{l_{m-1}}$, $r_{l_{n+1}} = r_N$, and $\Theta_{l_m}$ is the variable on the $m$th lens plane. As expected, stationary condition of the phase of the integrant in Eq. (7) reproduces the lens equation in the multi-lens system [10]

$$
\nabla_{\Theta_{l_{m'}}}\sum_{m=1}^{n}\left(\frac{r_{l_m}r_{l_{m+1}}}{2r_{l_m,l_{m+1}}}|\Theta_{l_m}-\Theta_{l_{m+1}}|^2-\psi(\Theta_{l_m})\right) = 0, \tag{9}
$$

for each $m' = 1, \cdots, n$, which hints at a way to the geometrical optics limit. The Gaussian approximation around a stationary solution yields the result in the geometrical optics limit (see also [6]).

## III. APPLICATION TO A SIMPLE CONFIGURATION

In this section we consider a simple case with two lenses. Fig. 2 shows the configuration: The Schwarzschild lenses with mass $M_1$ and $M_2$ are located at the radius $r_1$ and $r_2$, respectively. The model considered here is not general, because the source and the two lenses are arranged to be on a straight line. However, this simplification allows us to perform integration of Eq. (7) analytically. We start by rewriting Eq. (7)

$$
\begin{aligned}
F &= \frac{k}{2\pi i}\frac{r_1r_2}{r_{1,2}}\int d^2\Theta_1\exp\left[ik\left(\frac{r_1r_2}{2r_{1,2}}|\Theta_1-\Theta_2|^2-\psi(\Theta_1)\right)\right] \\
&\times \frac{k}{2\pi i}\frac{r_2r_3}{r_{2,3}}\int d^2\Theta_2\exp\left[ik\left(\frac{r_2r_3}{2r_{2,3}}|\Theta_2-\Theta_3|^2-\psi(\Theta_2)\right)\right]
\end{aligned}
\tag{10}
$$

with

$$
\psi(\Theta_j) = 4GM_j\ln\left(|\Theta_j|\right), \tag{11}
$$

for $j = 1, 2$, where $r_{i,j} = r_j - r_i$ and the possition of an observer is $(r_3, \Theta_3)$. From Eq. (10) we have

$$
\begin{aligned}
F &= \frac{k}{i}\frac{r_1r_2}{r_{1,2}}\int_0^\infty d\theta_1\theta_1^{1-4ikGM_1}\exp\left[ik\frac{r_1r_2}{2r_{1,2}}\left(\theta_1^2+\theta_2^2\right)\right]J_0\left(\frac{kr_1r_2}{r_{1,2}}\theta_1\theta_2\right) \\
&\times \frac{k}{i}\frac{r_2r_3}{r_{2,3}}\int_0^\infty d\theta_2\theta_2^{1-4ikGM_2}\exp\left[ik\frac{r_2r_3}{2r_{2,3}}\left(\theta_2^2+\theta_3^2\right)\right]J_0\left(\frac{kr_2r_3}{r_{2,3}}\theta_2\theta_3\right),
\end{aligned}
\tag{12}
$$

where $|\Theta_j| = \theta_j$ and $J_0(y)$ is the Bessel function of the first kind. Integration with respect to $\theta_1$ can be performed (see [10])

$$\begin{aligned} F &= e^{i\alpha} e^{\pi k G M_1} \Gamma(1-2ikGM_1) \frac{k}{i} \frac{r_2 r_3}{r_{2,3}} \int_0^\infty d\theta_2 \theta_2^{1-4ikGM_2} \\ &\times \exp\left[ ik \left( \frac{r_1 r_2}{2r_{1,2}} + \frac{r_2 r_3}{2r_{2,3}} \right) \theta_2^2 \right] {}_1F_1 \left( 1-2ikGM_1, 1\ ; \frac{-ikr_1 r_2}{2r_{1,2}} \theta_2^2 \right) J_0 \left( \frac{kr_2 r_3}{r_{2,3}} \theta_2 \theta_3 \right), \end{aligned} \tag{13}$$

where $\alpha$ is a real number which represents a constant phase and ${}_1F_1(a, b\ ; y)$ is the Kummer's function. With the use of the definition of the Bessel function

$$J_0(z) = \sum_{L=0}^{\infty} \frac{(-1)^L}{(L!)^2} \left( \frac{z}{2} \right)^{2L}, \tag{14}$$

we have (see [11])

$$\begin{aligned} F &= e^{i\alpha'} e^{\pi k G (M_1+M_2)} \Gamma(1-2ikGM_1)\ z \sum_{L=0}^{\infty} \frac{(-i)^L}{(L!)^2} \Gamma(1+L-2ikGM_2) \\ &\times (xz)^L {}_2F_1 \left( 1-2ikGM_1, 1+L-2ikGM_2, 1\ ; 1-z \right), \end{aligned} \tag{15}$$

where we defined

$$z = \frac{r_3(r_2-r_1)}{r_2(r_3-r_1)}, \tag{16}$$

$$x = \frac{kr_2 r_3 \theta_3^2}{2(r_3-r_2)}, \tag{17}$$

$\alpha'$ is a real constant and ${}_2F_1(a, b, c\ ; y)$ is the Hypergeometric function. In the limit $\theta_3 = 0$ ($x = 0$), Eq. (15) reduces to

$$\begin{aligned} F &= e^{i\alpha'} e^{\pi k G (M_1+M_2)} \Gamma(1-2ikGM_1) \Gamma(1-2ikGM_2) \\ &\times z\, {}_2F_1 \left( 1-2ikGM_1, 1-2ikGM_2, 1\ ; 1-z \right), \end{aligned} \tag{18}$$

and we have

$$|F|^2 = \frac{4\pi kGM_1}{1-e^{-4\pi kGM_1}} \frac{4\pi kGM_2}{1-e^{-4\pi kGM_2}} z^2 \left| {}_2F_1 \left( 1-2ikGM_1, 1-2ikGM_2, 1\ ; 1-z \right) \right|^2. \tag{19}$$

We consider the coincidence limit that the distance between the two lenses becomes zero, i.e., $r_1 = r_2$. In this case, from previous investigation (e.g., [10]), $F$ should be

$$\mathcal{F}(x) \equiv e^{\pi k G (M_1+M_2)} \Gamma(1-2ikGM_1) {}_1F_1 \left( 1-2ikG(M_1+M_2), 1\ ; -ix \right), \tag{20}$$

excepting a constant phase factor. We compare our result with Eq. (20). First, let us consider the limit $r_1 = r_2$, i.e., $z = 0$ of Eq. (15). Using the mathematical formula

$$\begin{aligned} &\Gamma(1-2ikGM_1)\Gamma(1-2ikGM_2) z\, {}_2F_1 \left( 1-2ikGM_1, 1-2ikGM_2, 1\ ; 1-z \right) \\ &= z^{2ikG(M_1+M_2)} \Gamma(1-2ikG(M_1+M_2)) \\ &\times {}_2F_1 \left( 2ikGM_1, 2ikGM_2, -1+2ikG(M_1+M_2)\ ; z \right) \\ &+ z \frac{\Gamma(1-2ikGM_1)\Gamma(1-2ikGM_2)\Gamma(-1+2ikG(M_1+M_2))}{\Gamma(2ikGM_1)\Gamma(2ikGM_2)} \\ &\times {}_2F_1 \left( 1-2ikGM_1, 1-2ikGM_2, 2-2ikG(M_1+M_2)\ ; z \right), \end{aligned} \tag{21}$$

in the case $\theta_3 = 0$ ($x = 0$), we can easily have

$$\lim_{z \to 0} |F|^2 = \frac{4\pi kG(M_1+M_2)}{1-e^{-4\pi kG(M_1+M_2)}} = |\mathcal{F}(0)|^2. \tag{22}$$

This is the expected result. Note that $|F|^2$ is regarded as the magnification factor. Fig. 3 plots $R = |F/\mathcal{F}(0)|$, as a function of $z$, where we fixed $kGM_1 = kGM_2 = 1$. Thus the maximum magnification depends on lens configuration significantly. Fig. 4 plots $R = |F/\mathcal{F}(x)|$ as a function of $x$ when fixing $z = 0.5$ and $kGM_1 = kGM_2 = 1$, which indicates that interference pattern also depends on lens configuration [12]. In these figures we have fixed $kGM_1 = kGM_2 = 1$, but result depends significantly on the parameters $kGM_1$, $kGM_2$ and $z$, as shown in Fig. 4, which suggests variety of wave effect depending on lens configuration.

## IV. SUMMARY AND CONCLUSION

In the present paper, we have presented a general formula to investigate wave effect in multi-lens system, which has been derived with the use of path integral approach. The formula is expressed in terms of integration with respect to variables of lens planes. It is difficult to perform the integration in general cases, but a system with two Schwarzschild lenses is an example for which the integration can be performed in an analytic way. The model of considered in the present paper is simplified and limited, however, it suggests variety of wave effect in multi-lens phenomenon. It is required to develop a numerical method to perform integration in general lens configuration in future work.

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[1] R. Takahashi and T. Nakamura, astro-ph/0305055
[2] N. Seto, astro-ph/0305605
[3] K. Yamamoto and K. Tsunoda, Phys. Rev. D 68, 041302(R) (2003)
[4] F. de Paolis, G. Ingross, A. A. Nucita, and A. Qadir, Astron. Astrophys. 394, 749 (2002)
[5] C. Baraldo, A. Hosoya, and T. T. Nakamura, Phys. Rev. D 59, 083001 (1999).
[6] T. T. Nakamura and S. Deguchi, Prog. Theor. Phys. Suppl. 133, 137 (1999)
[7] T. T. Nakamura, Phys. Rev. Lett. 80, 1138 (1998)
[8] S. Deguchi and W. D. Watson, Phys. Rev. D 34, 1708 (1986)
[9] S. Deguchi and W. D. Watson, Astrophys. J. 307, 30 (1986)
[10] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992)
[11] W. Magnus, F. Oberhettinger, R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Spriger-Verlag, Berlin, 1966)
[12] The computation was done with the Mathematica.

## APPENDIX A: DERIVATION OF EQ. (7)

In this Appendix, we review derivation of Eq. (7) from Eq. (6). We consider the configuration depicted as Fig. 1. The source is located at the origin of coordinate and the position of an observer is specified by $(r, \Theta) = (r_N, \Theta_N)$. The space between the source and the observer is discritized by $N - 1$ planes. The radius of each plane is labeled by $r_j$ for $j = 1, \cdots, N - 1$. Note that $r_N$ specifies the plane of the observer. $\Theta_j$ is (angle) variable on the $j$th plane. We consider the system with $n$ lenses, and assume that $m$th lens is located at the radius $r_{l_m}$. Assuming the validity of the thin lens approximation, we introduce the two dimensional potential

$$\psi(\Theta_{l_m}) = 2 \int_{r_l - \delta r}^{r_l + \delta r} dr U(r, \Theta), \tag{A1}$$

for $m = 1, \cdots, n$, respectively. In this case, the path integral formula (6) can be written as

$$F = \left[ \prod_{j=1}^{N-1} \int \frac{d^2\Theta_j}{A_j} \right] \exp \left[ ik \left( \epsilon \sum_{j=1}^{N-1} \frac{r_j r_{j+1}}{2} \left| \frac{\Theta_{j+1} - \Theta_j}{\epsilon} \right|^2 - \sum_{m=1}^{n} \psi(\Theta_{l_m}) \right) \right], \tag{A2}$$

where the normalization is chosen

$$A_j = \frac{2\pi i \epsilon}{k r_j r_{j+1}}, \tag{A3}$$

so that $F = 1$ in the limit $\psi = 0$. Then, Eq. (A2) is rephrased as

$$\begin{aligned} F &= \left[\prod_{j=1}^{l_1-1} \int \frac{d^2\Theta_j}{A_j}\right] \exp\left[ik\left(\epsilon \sum_{j=1}^{l_1-1} \frac{r_j r_{j+1}}{2} \left|\frac{\Theta_{j+1}-\Theta_j}{\epsilon}\right|^2\right)\right] \\ &\times \left[\prod_{j=l_1}^{l_2-1} \int \frac{d^2\Theta_j}{A_j}\right] \exp\left[ik\left(\epsilon \sum_{j=l_1}^{l_2-1} \frac{r_j r_{j+1}}{2} \left|\frac{\Theta_{j+1}-\Theta_j}{\epsilon}\right|^2 - \psi(\Theta_{l_1})\right)\right] \\ &\qquad\qquad\cdot \\ &\qquad\qquad\cdot \\ &\qquad\qquad\cdot \\ &\times \left[\prod_{j=l_n}^{N-1} \int \frac{d^2\Theta_j}{A_j}\right] \exp\left[ik\left(\epsilon \sum_{j=l_n}^{N-1} \frac{r_j r_{j+1}}{2} \left|\frac{\Theta_{j+1}-\Theta_j}{\epsilon}\right|^2 - \psi(\Theta_{l_n})\right)\right], \end{aligned} \tag{A4}$$

where $\epsilon$ is the separation between two neighboring planes. With the use of the following equality, which can be proven by the mathematical induction,

$$\begin{aligned} &\sum_{j=l_m}^{l_{m+1}-1} r_j r_{j+1} \left|\Theta_{j+1}-\Theta_j\right|^2 \\ &\qquad = \epsilon \frac{r_{l_m} r_{l_{m+1}}}{r_{l_{m+1}} - r_{l_m}} \left|\Theta_{l_{m+1}} - \Theta_{l_m}\right|^2 + \sum_{j=l_m+1}^{l_{m+1}-1} r_j^2 \frac{r_{j+1}-r_{l_m}}{r_j - r_{l_m}} \left|\Theta_j - u_{l_m,j}\right|^2 \end{aligned} \tag{A5}$$

with

$$u_{l_m,j} = \frac{r_{l_m}\Theta_{l_m} + (j-l_m) r_{j+1}\Theta_{j+1}}{j(r_{j+1}-r_{l_m})}, \tag{A6}$$

we have

$$\begin{aligned} F &= \int d^2\Theta_{l_1} \frac{k}{2\pi i} \frac{r_{l_1} r_{l_2}}{(r_{l_2}-r_{l_1})} \exp\left[ik\left(\frac{r_{l_1} r_{l_2}}{2(r_{l_2}-r_{l_1})} \left|\Theta_{l_2}-\Theta_{l_1}\right|^2 - \psi(\Theta_{l_1})\right)\right] \\ &\times \int d^2\Theta_{l_2} \frac{k}{2\pi i} \frac{r_{l_2} r_{l_3}}{(r_{l_3}-r_{l_2})} \exp\left[ik\left(\frac{r_{l_2} r_{l_3}}{2(r_{l_3}-r_{l_2})} \left|\Theta_{l_3}-\Theta_{l_2}\right|^2 - \psi(\Theta_{l_2})\right)\right] \\ &\qquad\qquad\cdot \\ &\qquad\qquad\cdot \\ &\qquad\qquad\cdot \\ &\times \int d^2\Theta_{l_n} \frac{k}{2\pi i} \frac{r_{l_n} r_N}{(r_N-r_{l_n})} \exp\left[ik\left(\frac{r_{l_n} r_N}{2(r_N-r_{l_n})} \left|\Theta_N-\Theta_{l_n}\right|^2 - \psi(\Theta_{l_n})\right)\right], \end{aligned} \tag{A7}$$

which is equivalent to Eq. (7)

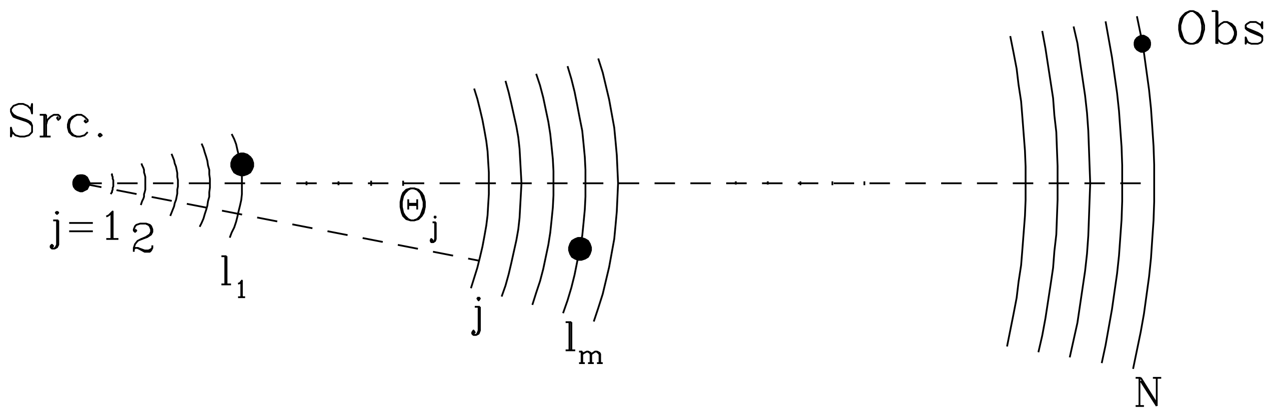


FIG. 1: Configuration of multi-lens system and coordinates for path integral formula.

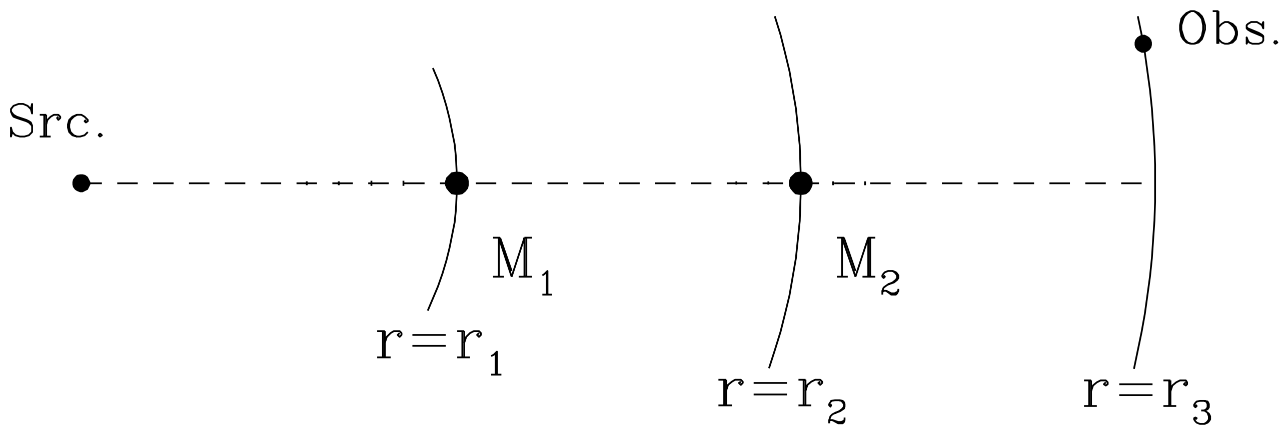


FIG. 2: Lensing system considered in section 3.

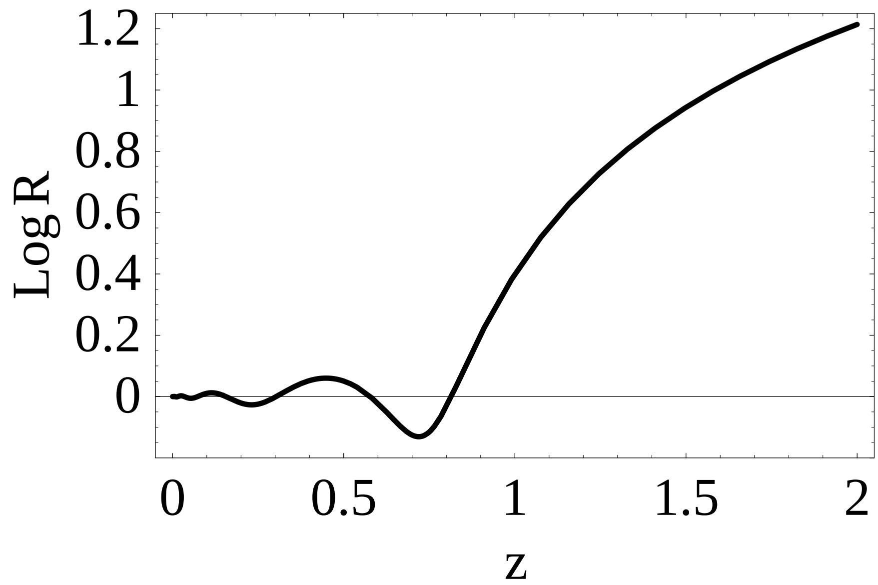


FIG. 3: $R = |F/\mathcal{F}(0)|$ as a function of $z$ with fixing $kGM_1 = kGM_2 = 1$.

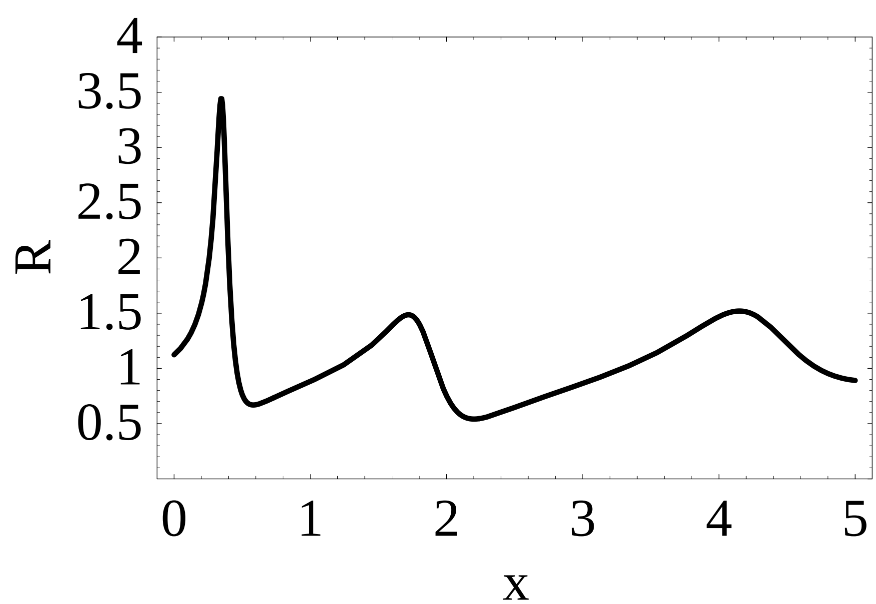


FIG. 4: $R = |F/\mathcal{F}(x)|$ as a function of $x$ with fixing $z = 0.5$ and $kGM_1 = kGM_2 = 1$.